Homework 8 Solutions

1. Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.

   (a) Find the eigenvalues of A. Show your work

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 0 & 2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} \\ &= -\lambda \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix} \\ &= -\lambda(2-\lambda)(3-\lambda) + 2(2-\lambda) \\ &= (2-\lambda)(-\lambda(3-\lambda)+2) \\ &= (2-\lambda)(\lambda^2 - 3\lambda + 2) \\ &= (2-\lambda)(\lambda-2)(\lambda-1) \\ &= -(\lambda-2)^2(\lambda-1) \end{aligned}$$

   Setting this to zero, we obtain the eigenvalues $\lambda = 1, 2$.

   (b) Determine the eigenvalues of A^3 without computing A^3. Explain your reasoning.

   The eigenvalues of A^3 are $1^3 = 1$ and $2^3 = 8$.

   (c) Determine the eigenvalues of A^T without computing A^T. Explain your reasoning.

   The eigenvalues of A^T are the same as the eigenvalues of A; hence they are $1, 2$.

   (d) Determine the eigenvalues of A^{-1} without computing A^{-1}. Explain your reasoning.?

   By problem 2 on Worksheet 10/31, the eigenvalues of A^{-1} are the recipricols of the eigenvalues of A; hence they are 1 and $\frac{1}{2}$.

2. Let $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ -2 & 1 & 5 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

(a) Determine the eigenvalues of A. Explain your reasoning.

Since A is triangular, its eigenvalues are the diagonal entries of A, which are -1, 3, and 5.

(b) Find a basis for the eigenspaces corresponding to the eigenvalues you found in part (a). Show your work.

$\underline{E_{-1}(A)}$ We must determine $N(A+I)$, which we do by solving the system $(A+I)\mathbf{x} = \mathbf{0}$. Reducing $A + I$ to RREF yields

$$A + I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ -2 & 1 & 6 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence x_4 is the free variable and we have the general solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus $E_{-1}(A)$ has basis $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\underline{E_3(A)}$ We must determine $N(A-3I)$, which we do by solving the system $(A-3I)\mathbf{x} = \mathbf{0}$. Reducing $A - 3I$ to RREF yields

$$A - 3I = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -2 & 1 & 2 & 0 \\ 1 & 0 & 0 & -4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence x_3 is the free variable and we have the general solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Thus $E_3(A)$ has basis $\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

$E_5(A)$ We must determine $N(A-5I)$, which we do by solving the system $(A-5I)\mathbf{x} = \mathbf{0}$. Reducing $A - 5I$ to RREF yields

$$A - 5I = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & -6 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence x_3 is the free variable and we have the general solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus $E_3(A)$ has basis $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(c) What are the algebraic and geometric multiplicities of the eigenvalues? Explain your reasoning.

The characteristic polynomial is $\det(A - \lambda I) = (\lambda+1)^2(\lambda-3)(\lambda-5)$ (since $A - \lambda I$ is triangular). Hence -1 has algebraic multiplicity 2, which 2 and 5 have algebraic multiplicity 1.

The geometric multiplicities are the dimensions of the eigenspaces. By part (b), we see that the eigenspaces all have dimension 1; hence geometric multiplicities are all 1.

(d) Is A diagonalizable? Explain your reasoning.

A is not diagonalizable because there do not exist four linearly independent eigenvectors. This follows from the fact that the eigenspaces are all 1-dimensional, implying that there are only three linearly independent eigenvectors.

3. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. Recall that the eigenspace corresponding to λ is $E_\lambda(A) = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \lambda\mathbf{v}\}$. In class, we saw that $E_\lambda(A) = N(A-\lambda I)$,

which automatically implies it is a subspace of $\mathbb{R}^n$. Let's prove this a different way. Show that $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$ using the definition of subspaces.

- Since $A\mathbf{0} = \mathbf{0} = \lambda\mathbf{0}$, we have $\mathbf{0} \in E_\lambda(A)$.
- Let $\mathbf{v}, \mathbf{w} \in E_\lambda(A)$. Then

$$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w} = \lambda\mathbf{v} + \lambda\mathbf{w} = \lambda(\mathbf{v} + \mathbf{w}).$$

  Hence $\mathbf{v} + \mathbf{w} \in E_\lambda(A)$.
- Let $\mathbf{v} \in E_\lambda(A)$ and $c \in \mathbb{R}$. Then

$$A(c\mathbf{v}) = cA\mathbf{v} = c(\lambda\mathbf{v}) = \lambda(c\mathbf{v}).$$

  Hence $c\mathbf{v} \in E_\lambda(A)$.

4. Show that A has an eigenvalue of 0 if and only if A is not invertible.

($\Rightarrow$) Suppose A has an eigenvalue of 0 and let $\mathbf{v}$ be a corresponding eigenvector. Then $A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$. Since $\mathbf{v}$ is nonzero, we have that $A\mathbf{x} = \mathbf{0}$ has a nonzero solution (i.e. $\mathbf{x} = \mathbf{v}$). This implies that A is not invertible.

($\Leftarrow$) Suppose that A is invertible. Then $A\mathbf{x} = \mathbf{0}$ has a nonzero solution $\mathbf{v}$. Hence $A\mathbf{v} = \mathbf{0} = 0\mathbf{v}$ and so 0 is an eigenvalue of A.

5. Let A be an $n \times n$ matrix with n eigenvalues (counted with multiplicity). Use the characteristic polynomial to show that the determinant of A is the product of the eigenvalues of A.

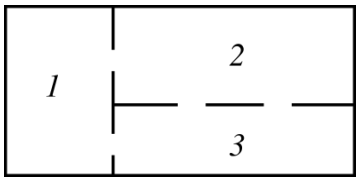
$\det(A - \lambda I)$ is a degree n polynomial. We first show that the leading coefficient (the coefficient in front of λ^n is $(-1)^n$. We show this by computing det using coactor expansions repeatedly along the first row:

$$\begin{aligned}
\begin{vmatrix} a_{11} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix} &= (a_{11} - \lambda) \begin{vmatrix} a_{22} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix} + \text{terms of degree less than } n \\
&= (a_{11} - \lambda)(a_{22} - \lambda) \begin{vmatrix} a_{33} - \lambda & & \\ & \ddots & \\ & & a_{nn} - \lambda \end{vmatrix} + \text{terms of degree less than } n \\
&= (a_{11} - \lambda) \cdots (a_{nn} - \lambda) + \text{terms of degree less than } n
\end{aligned}$$

Multiplying out the first product, we see that the coefficient in front of the λ^n term is $(-1)^n$.

Now, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A. Then characteristic polynomial $\det(A - \lambda I)$ can be factored into $\det(A - \lambda I) = (-1)^n(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$. Setting $\lambda = 0$ yields $\det(A) = \lambda_1 \cdots \lambda_n$.

6. A scientist places a mouse in a box with 3 compartments, as shown below.



Notice that there are two doors between compartments 2 and 3, there is one door between compartments 1 and 2, and there is one door between compartments 1 and 3. The mouse has been trained to select a door at random and move through it whenever a bell is rung. For example, when the mouse is in compartment 2, there is a 1/3 probability that it will move to compartment 1, a 2/3 probability that it will move to compartment 3, and a zero probability that the it will remain in compartment 2. Let M be the matrix whose (i, j)th entry contains the probability that the mouse will move to compartment i given that it is in compartment j.

(a) Write down M and show that it is a regular Markov matrix.

$M = \begin{bmatrix} 0 & 1/3 & 1/3 \\ 1/2 & 0 & 2/3 \\ 1/2 & 2/3 & 0 \end{bmatrix}$. Since the columns are all probability vectors, M is a Markov matrix. Since $P^2 = \begin{bmatrix} 1/3 & 2/9 & 2/9 \\ 1/3 & 11/18 & 1/6 \\ 1/3 & 1/6 & 11/18 \end{bmatrix}$ has all positive entries, M is regular.

(b) Find the steady state vector of M. Show your work.

First we determine $E_1(M) = N(M - I)$. Hence we must solve $(M - I)\mathbf{x} = \mathbf{0}$. Reducing $M - I$ to RREF we get:

$$M - I = \begin{bmatrix} -1 & 1/3 & 1/3 \\ 1/2 & -1 & 2/3 \\ 1/2 & 2/3 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence x_3 is the free variable and the general solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/3x_3 \\ x_3 \\ x_3 \end{bmatrix}$.

The steady state vector is a probability vector. Hence we must choose an eigenvector whose entries add to 1. Thus: $2/3x_3 + x_3 + x_3 = 1$, or $x_3 = 3/8$. Thus the steady state vector is $\mathbf{q} = \begin{bmatrix} 1/4 \\ 3/8 \\ 3/8 \end{bmatrix}$.

(c) Suppose the mouse is placed in compartment i. Then at the start of the experiment, the initial probability vector is $\mathbf{x}_0 = \mathbf{e}_i$. This vector tells us that the probability that the mouse is in compartment i is 1 and the probability that the mouse is in one of the other compartments is 0. Consider the Markov chain $\mathbf{x}_k =^k \mathbf{x}_0$. After the kth ring of the bell, the jth entry of $\mathbf{x}_k$ is the probability that the mouse is in the jth compartment (where $j = 1, 2,$ or 3). Without computation, find $\lim_{k\to\infty} \mathbf{x}_k$. Explain your reasoning.

Since M is a regular Markov matrix, every Markov chain will converge to the steady state vector, namely $\begin{bmatrix} 1/4 \\ 3/8 \\ 3/8 \end{bmatrix}$.

(d) Assuming that the bell is rung in regular intervals, what does the steady-state vector tell you about how much time the mouse spends in each compartment, in the long run?

In the long run, the mouse will spend 25% of its time in compartment 1, 37.5% of its time in compartment 2, and 37.5% of its time in compartment 3.

7. This problem considers a Markov matrix that is not regular. Let $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(a) Diagonalize M by hand; that is, find matrices P and D such that $M = PDP^{-1}$.

We first find the eigenvalues of M. First, using a cofactor expansion along the first row, we find:

$$\det(M - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1).$$

Setting the characteristic polynomial equal to zero and solving, we get $\lambda = 1, -1$. Next we find bases for the eigenspaces.

$\underline{E_1(M)}$ We must determine $N(M-I)$, which we do by solving the system $(M-I)\mathbf{x} = \mathbf{0}$. Reducing $M - I$ to RREF yields

$$M - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence x_1 and x_3 are free variables and we have the general solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Thus $E_1(M)$ has basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$\underline{E_{-1}(M)}$ We must determine $N(M+I)$, which we do by solving the system $(M+I)\mathbf{x} = \mathbf{0}$. Reducing $M - I$ to RREF yields

$$M + I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence x_3 is the free variable and we have the general solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Thus $E_1(M)$ has basis $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

Thus we have $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

(b) Use part (a) to write down a formula for M^k. Deduce that M is not regular.

$$\begin{aligned} M^k = PD^KP^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & .5 - .5(-1)^{k+1} & .5 + .5(-1)^{k+1} \\ 0 & .5 - .5(-1)^k & .5 + .5(-1)^k \end{bmatrix}. \end{aligned}$$

Since M^k has zero entries for all k, M is not regular.

(c) Describe all steady state vectors of M. Show your work.

The steady state vectors are eigenvectors corresponding to 1 whose entries add to 1. By part (b) The eigenvalues corresponding to 1 are of the form $\begin{bmatrix} a \\ b \\ b \end{bmatrix}$, where $a, b \in \mathbb{R}$. To ensure the entries sum to 1, we must require $a + 2b = 1$ or $b = \frac{1-a}{2}$. Hence the steady state vectors form the set

$$\left\{ \begin{bmatrix} a \\ \frac{1-a}{2} \\ \frac{1-a}{2} \end{bmatrix} \;\middle|\; 0 \leq a \leq 1 \right\}.$$

(d) For the following initial state vectors, determine to which steady state vector (if any) the Markov chain $\mathbf{x}_k = M^k \mathbf{x}_0$ converges. Explain your reasoning.

$$(i) \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad (ii) \quad \mathbf{x}_0 = \begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix} \qquad (iii) \quad \mathbf{x}_0 = \begin{bmatrix} 0 \\ .5 \\ .5 \end{bmatrix}$$

(i): Since $\mathbf{x}_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .5 - .5(-1)^{k+1} & .5 + .5(-1)^{k+1} \\ 0 & .5 - .5(-1)^k & .5 + .5(-1)^k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, we have $\mathbf{x}_k \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ as $k \to \infty$.

(ii): Since $\mathbf{x}_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .5 - .5(-1)^{k+1} & .5 + .5(-1)^{k+1} \\ 0 & .5 - .5(-1)^k & .5 + .5(-1)^k \end{bmatrix} \begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .25 - .25(-1)^{k+1} \\ .25 - .25(-1)^k \end{bmatrix}$, we have that $\mathbf{x}_k$ alternates between $\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} .5 \\ 0 \\ .5 \end{bmatrix}$ as $k \to \infty$. Hence this Markov chain does not converge.

(i): Since $\mathbf{x}_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .5 - .5(-1)^{k+1} & .5 + .5(-1)^{k+1} \\ 0 & .5 - .5(-1)^k & .5 + .5(-1)^k \end{bmatrix} \begin{bmatrix} 0 \\ .5 \\ .5 \end{bmatrix} = \begin{bmatrix} 0 \\ .5 \\ .5 \end{bmatrix}$, we have $\mathbf{x}_k \to \begin{bmatrix} 0 \\ .5 \\ .5 \end{bmatrix}$ as $k \to \infty$.